Chapter 10
Operators of the scalar Klein Gordon field
–
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Understanding Relativistic Quantum Field Theory

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Chapter 10

Operators of the scalar Klein Gordon field

10.1 The scalar charge-current density

The Klein Gordon equation is generally associated with a scalar field, a field which values do not change under Lorentz transform, but only changes coordinates. We must realize that this is a special case however. Often more complex fields will have components which individually all obey the Klein Gordon equation, for instance the Dirac field which is a bi-spinor. The relevance of the Klein Gordon equation goes well beyond the scalar field with which it is generally associated. We can express a lorentz transform of the scalar field ψ as.

$$\psi' \quad = \quad \psi(\Lambda^{-1}x) \tag{10.1}$$

Where Λ is a 4x4 matrix representing the Lorentz transform. Note that Λ^{-1}, the reversed transform, is used here. If the original field is concentrated let's say at a position x_{org}, then this coordinate will be transformed by Λ to a new position x_{new}. Working in the new reference frame and looking at x_{new} we should obtain the value given by the original function $\psi(x_{org})$. Therefor we need to reverse transform our coordinate x_{new} back to x_{org}.

If we start with the particle's field in its rest frame and we transform to a frame in which it moves with a velocity v, then the volume containing the field will Lorentz contract by a factor γ. This means that the probability densities in this volume should increase by a factor γ also. We have however defined our field ψ defined as a scalar field which does not changes its values under transform. This means that $\psi^*\psi$ can not represent the probability density of the scalar field as it does in case of the Schrödinger equation.

We need a function which at one hand is proportional to $\psi^*\psi$, but at the other hand also increases with gamma. For the Schrödinger equation we have the expressions for the probability flux which actually do transform correct. If we denote the probability with P^o and the probability flux with P^i then we have a relativistically correct transforming 4-vector. The probability flux for the Schrödinger field is given by.

$$\mathcal{P}^i \quad = \quad -\frac{i\hbar}{2m}\left(\psi^*\frac{\partial\psi}{\partial x^i} - \frac{\partial\psi^*}{\partial x^i}\psi \right) \tag{10.2}$$

This expression produces a real values per definition and it is instructive to substitute ψ with a complex plane wave eigen-function.

$$\psi \quad = \quad \exp\Big(-iEt/\hbar + i\vec{p}\cdot\vec{x}/\hbar \Big) \tag{10.3}$$

We see that only the phase-change rate of ψ ends up in the probability flux, while the imaginary part, representing the magnitude change of ψ is eliminated by the subtraction.

We can now see that we get the required expression for the probability density P^o if we take the derivatives in time x^o=ct.

$$\mathcal{P}^o \quad = \quad \frac{i\hbar}{2m}\left(\psi^*\frac{\partial\psi}{\partial x^o} - \frac{\partial\psi^*}{\partial x^o}\psi \right) \tag{10.4}$$

The operator (the time-derivative in this case) is applied symmetrically to both ψ and ψ^* and the sign is adopted to select the imaginary part of the exponent representing the phase change rate in time.

Since the total probability integrated over space is 1, and the total charge should be equal to e, we can use the above expressions as the charge-current density of the scalar Klein Gordon field in the Pauli-Weisskopf interpretation.

$$\mathcal{J}^\mu \quad = \quad \frac{ie\hbar}{2m}\left(\psi^*\frac{\partial\psi}{\partial x_\mu} - \frac{\partial\psi^*}{\partial x_\mu}\psi \right) \tag{10.5}$$

The Pauli-Weisskopf interpretation easies the 'negative probability' issues associated with anti-particles given by.

$$\psi \quad = \quad \exp\Big(+iEt/\hbar + i\vec{p}\cdot\vec{x}/\hbar \Big) \tag{10.6}$$

In this case the sign of the charge-density changes indicating the opposite charge of the anti-particle.

For a single particle wave-function we expect unitarity. This leads to the requirement that the four divergence of the charge-current density vanishes. We need the continuity relation.

$$\frac{\partial \mathcal{J}^\mu}{\partial x^\mu} = \frac{ie\hbar}{2m}\left(\frac{\partial\psi^*}{\partial x^\mu}\frac{\partial\psi}{\partial x_\mu} - (\Box\psi^*)\psi + \psi^*\Box\psi - \frac{\partial\psi^*}{\partial x_\mu}\frac{\partial\psi}{\partial x^\mu}\right) = 0 \quad (10.7)$$

That is. The Total current leaving an infinitesimal volume-element minus the current entering it gives the decrease of charge per unit of time in the volume element. If ϕ obeys the Klein Gordon equation then we can replace this expression by:

$$\frac{\partial \mathcal{J}^\mu}{\partial x^\mu} = \frac{ie\hbar}{2m}\left(\frac{\partial\psi^*}{\partial x^\mu}\frac{\partial\psi}{\partial x_\mu} - m^2\psi^*\psi + \psi^* m^2\psi - \frac{\partial\psi^*}{\partial x_\mu}\frac{\partial\psi}{\partial x^\mu}\right) = 0 \quad (10.8)$$

Which trivially vanishes. This means that J^μ is a conserved current and that unitarity is guaranteed.

10.2 Klein Gordon charge-current density operators

In the previous section we did see that we can define the electric charge-density operator and the current-density operator for the Klein Gordon field as follows.

$$\tilde{\mathsf{J}}^o = \frac{ie\hbar}{2m}\frac{\partial}{\partial x^o}, \qquad \tilde{\mathsf{J}}^i = -\frac{ie\hbar}{2m}\frac{\partial}{\partial x^i} \quad (10.9)$$

Recall that $j^o = c\rho$ as being part of a four-vector. For this we need to define the way these operators act on the field ψ as.

$$\begin{aligned} \mathsf{J}^o &= +\psi^* \overleftrightarrow{\tilde{\mathsf{J}}^o}\, \psi &&= \left(\psi^*\, \tilde{\mathsf{J}}^o\, \psi \;+\; \psi\, \tilde{\mathsf{J}}^{o*}\, \psi^*\right) \\ \mathsf{J}^i &= -\psi^* \overleftrightarrow{\tilde{\mathsf{J}}^i}\, \psi &&= -\left(\psi^*\, \tilde{\mathsf{J}}^i\, \psi \;+\; \psi\, \tilde{\mathsf{J}}^{i*}\, \psi^*\right) \end{aligned} \quad (10.10)$$

The operators act symmetrically on both ψ and ψ^*. The practical reason is that the individual terms can contain imaginary parts and we expect the end result to be real. The imaginary parts cancel if we let the operator act in the above symmetric way.

The imaginary terms occur as soon as ψ is not a plane wave solution anymore, when its magnitude changes with time or space, for instance because it is localized.

The imaginary parts also vanish if we integrate the terms over space.

$$\mathfrak{Im}\left\{\int dx^3 \Big(\psi^*\ \tilde{J}^o\ \psi\Big)\ \right\} \quad = \quad \mathfrak{Im}\left\{\int dx^3 \Big(\psi\ \tilde{J}^{o*}\psi^*\Big)\ \right\} \quad = \quad 0 \quad (10.11)$$

They cancel because any variation in the magnitude of ψ can be seen as the result of interference between plane waves which do have constant magnitudes them self. Any interference between plane waves is a sinusoidal function which vanishes after being integrated over space.

The fact that we want to evaluate quantities locally instead of globally is what leads us to the expressions of (10.10)

10.3 General form for the local operator application

The charge-current density operators suggest that a more general form for operators should be.

$$\mathcal{O} \quad = \quad \psi^* \overleftrightarrow{\tilde{\mathcal{O}}}\ \psi \quad = \quad \Big(\ \psi^*\tilde{\mathcal{O}}\ \psi \quad + \quad \psi\ \tilde{\mathcal{O}}^*\psi^*\ \Big) \qquad (10.12)$$

If the field ψ is locally described by an exponent with an arbitrary amplitude part $a(x^\mu)$ and an arbitrary phase $\phi(x^\mu)$ like this:

$$\phi \quad = \quad \exp\Big(\ a\,(x^\mu) \ \ + \ i\phi\,(x^\mu)\ \Big) \qquad (10.13)$$

Then the general operator form of (10.12) guarantees that the change in amplitude doesn't play a role. Only the phase change rates of ψ do. There is a good reason to assume that for an important class of operators only the phase change rates determine the quantities associated with the operators.

For instance if we consider a static, typically bound, solution at rest without internal momentum then the phase change rate in time determines

the energy. Changing reference frames will transform this energy into momentum via the Lorentz transform. The class of local quantities for which we want to have the operators are typically based on the relativistic four-momentum.

The local quantities are densities and generally this is where the local amplitude of the wave-function comes in, in the form of the conjugate product $\psi^*\psi$ adapted by a factor γ due to the Lorentz contraction. The latter is specific for the *scalar* Klein Gordon field because the value of ϕ itself, being a Lorentz scalar, does not transform.

We will find that (10.12) is not the most general form. The problem arises when the operator contains higher order/ mixed derivatives. In these cases the phase and amplitude components do mix up again and we have to study the results carefully to find the correct way to apply the operators in these cases. This is the price we pay for representing a field with both phase and amplitude by complex numbers.

10.4 The time derivative Hamiltonian

The use of the time derivative Hamiltonian is often used to derive new operators from existing operators. More specifically, the operators which give the time derivatives of the quantities produced by existing operators.

$$i\hbar\frac{\partial}{\partial t} \quad = \quad \tilde{H} \tag{10.14}$$

The commutation of the operator with the Hamiltonian is used to obtain the time derivative of the observable.

$$\int dx^3 \; \frac{\partial}{\partial t}\Big(\; \psi^*\tilde{O} \; \psi \; \Big) \quad = \quad \frac{1}{i\hbar}\int dx^3 \; \psi^* \Big[\; \tilde{O}, \tilde{H} \; \Big]\psi \tag{10.15}$$

Although taking the time derivative directly is trivial, applying $\tilde{H}$ is generally much more interesting since it is an alternative representation which should lead to the same results. Often $\tilde{H}$ doesn't contain any time-derivative at all but obtains the same result by other means.

Naively applying $[\tilde{\mathcal{O}}, \tilde{H}]$ however leads easily to erroneous expressions, especially when we want them to be locally meaningful rather then globally after taking the integral over all space. We will therefor review here the conditions under which we are allowed to use the commutator with the Hamiltonian.

The time derivative of the local density of a quantity is given by.

$$\frac{\partial}{\partial t}\Big(\psi^* \tilde{\mathcal{O}}\, \psi \Big) \;=\; \psi^* \tilde{\mathcal{O}}\, \frac{\partial}{\partial t}\Big(\psi\Big) \;+\; \psi^* \frac{\partial}{\partial t}\Big(\tilde{\mathcal{O}}\Big)\psi \;+\; \frac{\partial}{\partial t}\Big(\psi^*\Big)\tilde{\mathcal{O}}\psi \qquad (10.16)$$

The more general operators do not change with time them shelf. This eliminates the middle term at the right hand side. Substituting the time derivatives with $\tilde{H}$ gives us.

$$\frac{\partial}{\partial t}\Big(\psi^* \tilde{\mathcal{O}}\, \psi \Big) \;=\; \frac{1}{i\hbar}\Big[\, \psi^* \tilde{\mathcal{O}}\Big(\tilde{H}\psi\Big) \;-\; \Big(\tilde{H}^*\psi^*\Big)\tilde{\mathcal{O}}\psi \,\Big] \qquad (10.17)$$

Where we have used $\tilde{H}^* = -\tilde{H}$ in the second occurrence of $\tilde{H}$ at the right hand side since it is Hermitic. We see that the correctness of the use of the commutator of the Hamiltonian as a time derivative depends on the presumption of.

$$\Big(\tilde{H}^*\psi^*\Big)\tilde{\mathcal{O}}\psi \;\overset{?}{=}\; \psi^* \tilde{H}\tilde{\mathcal{O}}\,\psi \qquad (10.18)$$

This assumption however can be made only in a limited number of cases, even if the operator $\tilde{\mathcal{O}}$ is unity! The magnitude of the field $|\psi|$ has to be static for the expression below to be valid. ($\partial|\psi|/\partial_t = 0$)

$$(\tilde{H}^*\psi^*)\psi \;=\; \psi^* \tilde{H}\,\psi \qquad (10.19)$$

The above identity is not true if the time derivative of ψ isn't purely imaginary. This condition is relaxed however if we integrate over all space.

$$\int dx^3\ \tilde{H}^*\psi^*\psi \;=\; \int dx^3\ \psi^* \tilde{H}\,\psi \qquad (10.20)$$

This identity is valid in general for linear $\tilde{H}$. In this case ϕ may be an arbitrary superposition of plane waves. The sinusoidal interference terms between the plane waves integrate away.

$$\begin{aligned} & \int dx \left(\psi_1 + \psi_2 + \psi_3 ... \right)^* \left(\psi_1 + \psi_2 + \psi_3 ... \right) \\ = & \quad \int dx^3 \left(\psi_1^* \psi_1 + \psi_2^* \psi_2 + \psi_3^* \psi_3 + ... \right) \end{aligned} \tag{10.21}$$

The complex conjugate product of the superposition is equal to the sum of the conjugate products of the individual components.

$$\psi \quad = \quad \exp\Big(\; a\,(x^\mu) \;\; + \;\; i\phi\,(x^\mu) \; \Big) \tag{10.22}$$

10.5 The position operator

For a position operator we can simply define a center of gravity operator. Starting from the charge-density operator, removing the charge from the normalization and multiplying with the position x^i

$$\tilde{\mathtt{X}}^i \quad = \quad \frac{i\hbar}{2mc}\, x^i\, \frac{\partial}{\partial x^o} \tag{10.23}$$

The operator is applied in the general form.

$$\mathtt{X}^i \quad = \quad \psi^* \; \overleftrightarrow{\tilde{\mathtt{X}}^i} \; \psi \quad = \quad \Big(\; \psi^* \; \tilde{\mathtt{X}}^i \; \psi \quad + \quad \psi \; \tilde{\mathtt{X}}^{i*} \, \psi^* \; \Big) \tag{10.24}$$

The density is compressed by a factor γ (higher by a factor γ) due to Lorentz contraction. It is the time derivative applied anti-symmetrically on ψ and ψ^* which produces this factor γ. The phase change rate in time is proportional to the energy $E = \gamma mc^2$.

$$\mathtt{X}^i \quad = \quad x^i \gamma\, \psi^* \psi \tag{10.25}$$

The integral over space gives the average position, the center of gravity.

$$x^i_{avg} \quad = \quad \int dx^3 \;\, \psi^* \;\, \overleftrightarrow{\tilde{\mathtt{X}}^i} \;\, \psi \quad = \quad \int dx^3 \;\, x^i \, \gamma \, \psi^* \psi \tag{10.26}$$

10.6 The velocity operator

We would like, as in non-relativistic quantum mechanics, to obtain the velocity operator from the position operator via a commutation with the Hamiltonian.

$$\tilde{\mathtt{V}}^i \;=\; \frac{1}{i\hbar}\Big[\,\tilde{\mathtt{X}}^i\;,\;\tilde{H}\,\Big] \tag{10.27}$$

However the Klein Gordon equation doesn't explicitly contain the first derivative in time but the second order instead which gives us the square of the Hamiltonian.

$$\tilde{H}^2 \;=\; -\,\hbar^2\frac{\partial^2}{\partial t^2}\psi \;=\; \Big(\,-\,\hbar^2c^2\nabla^2\psi\;+\;m^2c^4\,\Big)\psi \tag{10.28}$$

One way to obtain the Hamiltonian is via a derivation against itself.

$$\tilde{H} \;=\; \frac{1}{2}\frac{\partial}{\partial\tilde{H}}\Big(\,-\,\hbar^2c^2\nabla^2\psi\;+\;m^2c^4\,\Big)\psi \tag{10.29}$$

and using the time derivative $i\hbar\partial_t$ for $\tilde{H}$ at the righthand side. The rationale for this is that many operators, like for instance the position operator depend on the density which involves a time derivative.

The only non-commuting part of the Hamiltonian with the position operator is the second order derivative of the corresponding axis, so.

$$\frac{1}{i\hbar}\left[\,\frac{i\hbar}{2mc}\,x\,\frac{\partial}{\partial x^o}\;\;,\;-\,\hbar^2c^2\frac{\partial^2}{\partial x^2}\,\right] \;=\; \frac{\hbar^2c}{m}\,\frac{\partial}{\partial x}\,\frac{\partial}{\partial x^o} \tag{10.30}$$

The time derivative in $\tilde{X}$ was handled just like a constant here. We obtain the velocity operator after applying the derivative with regard to the Hamiltonian.

$$\tilde{\mathtt{V}}^i \;=\; \frac{1}{2}\frac{\partial}{\partial\tilde{H}}\left(\,\frac{\hbar^2c}{m}\,\frac{\partial}{\partial x}\,\frac{\partial}{\partial x^o}\,\right) \tag{10.31}$$

As said, to do this we define the Hamiltonian $\tilde{H}$ as the time derivative here. This then removes the time derivative from the expression and we

obtain the velocity operator. Which turns out to be proportional to the current-density operator in (10.9).

$$\tilde{\mathtt{V}}^i \;=\; -\frac{i\hbar}{2m}\frac{\partial}{\partial x^i} \tag{10.32}$$

Applying the operator $\tilde{V}^i$ on the Klein Gordon field in the standard way,

$$\mathtt{V}^i \;=\; \psi^* \;\overleftrightarrow{\tilde{\mathtt{V}}^i}\; \psi \;=\; \Big(\; \psi^*\; \tilde{\mathtt{V}}^i\; \psi \;+\; \psi\; \tilde{\mathtt{V}}^{i*} \psi^* \;\Big) \tag{10.33}$$

gives us V^i, the velocity-density of the Klein Gordon field ψ.

$$\mathtt{V}^i \;=\; -\frac{i\hbar}{2m}\left(\; \psi^*\frac{\partial\psi}{\partial x^i} \;-\; \frac{\partial\psi^*}{\partial x^i}\psi \;\right) \tag{10.34}$$

If we compare this with (10.5), for the current density, we see that they are equal up to the value e for the electric charge.

$$j^i \;=\; e\mathtt{V}^i \tag{10.35}$$

Which justifies the term velocity *density* for $\mathtt{V}^i$. The integral of $\mathtt{V}^i$ over all space gives us the velocity of the particle, or more accurate, the spatial average velocity.

When we apply the velocity operator on the Klein Gordon field ψ expressed in a form which separates amplitude and phase.

$$\psi \;=\; \exp(\; a + i\phi \;) \tag{10.36}$$

We obtain the following expression for the velocity density.

$$\mathtt{V}^i \;=\; \frac{\hbar}{m}\frac{\partial\phi}{\partial x^i}\;\psi^*\psi \tag{10.37}$$

Where the only dependence on the amplitude component a is found in $\psi^*\psi$. The subtraction in (10.34) has eliminated the $\partial a/\partial x$ terms, so the velocity

density does not depend on the derivatives of a, which is indeed what we want. The spatial phase change rate is proportional to the momentum.

$$\frac{\partial \phi}{\partial x^i} \;=\; \frac{p}{\hbar} \;=\; \frac{\gamma m v^i}{\hbar} \tag{10.38}$$

So we can express the velocity density $\mathtt{V}^i$ in a form which uses the local velocity v^i and the corresponding local γ

$$\mathtt{V}^i \;=\; \gamma v^i\, \psi^* \psi \tag{10.39}$$

We have defined the Klein Gordon field ψ as a Lorentz scalar and thus the term $\psi^*\psi$ does no change under Lorentz transform. The total volume which contains the field however does change. It undergoes a Lorentz contraction by a factor γ. This means that we have the following expression for the unitarity of the field.

$$\int \gamma\, \psi^* \psi \; dx^3 \;=\; 1 \tag{10.40}$$

Where the local γ compensates the local Lorentz contraction. We can now show that the integral over all space indeed leads to the (spatial average) velocity of the field.

$$v^i_{avg} \;=\; \int \mathtt{V}^i \; dx^3 \;=\; \int v^i\, \gamma\, \psi^* \psi \; dx^3 \tag{10.41}$$

or expressed using (10.34)

$$v^i_{avg} \;=\; -\frac{i\hbar}{2m} \int \left(\psi^* \frac{\partial \psi}{\partial x^i} \;-\; \frac{\partial \psi^*}{\partial x^i} \psi \right) dx^3 \tag{10.42}$$

We have established a way to obtain a local velocity v^i, with a corresponding local γ, from the Klein Gordon field ψ using the velocity density operator $\tilde{\mathtt{V}}^i$.

The average velocity v^i_{avg} is obtained by averaging the local velocity v^i over space using the relativistic effective density $\gamma\, \psi^* \psi$.

$$\gamma v^i \;\;=\;\; -\frac{i\hbar}{2m}\left(\frac{1}{\psi}\frac{\partial\psi}{\partial x^i} \;-\; \frac{1}{\psi^*}\frac{\partial\psi^*}{\partial x^i}\right) \tag{10.43}$$

The local γ can be obtained from the three components of γv^i via the standard relativistic equation.

$$\gamma \;=\; \cosh(\sinh|p|) = \cosh\left(\sinh^{-1}\sqrt{(\gamma v_x)^2+(\gamma v_y)^2+(\gamma v_z)^2}\right) \tag{10.44}$$

Nevertheless we need to be aware that what we call the local $\vec{v}$ and local γ are in fact average values in case of superposition and the more the wave-packet is localized the higher is the amount of superposition and the wider the range of momenta.

The local velocity we obtain is the same velocity we obtain if we use the current density divided by the charge density. Classically, in the case of a current carrying electric wire, this is called the *average drift* velocity. In an actual wire there are many charges moving at all kinds of different speeds.

We can however carry out the electromagnetic calculations for the Lorentz force *and* transform them correctly from one to another reference frames only with the knowledge of these averaged current and the charge, without the specific knowledge of the speed of all the individual charges. (See the appendix on the *Magnetism as a relativistic side effect of electrostatics* where the mechanism behind this is explained)

10.7 The acceleration operator

Having derived the velocity operator, leading to a standard result, we like to go one step further here: We want to obtain the *acceleration* operator. We do so by applying the Hamiltonian twice on the position operator.

$$\tilde{\mathtt{A}}^i \;\;=\;\; -\frac{1}{\hbar^2}\Big[\,\Big[\,\tilde{\mathtt{X}}^i\,,\;\tilde{H}\,\Big]\,,\;\tilde{H}\,\Big] \;\;=\;\; -\frac{1}{\hbar^2}\Big[\,\tilde{\mathtt{X}}^i\,,\;\tilde{H}^2\,\Big] \tag{10.45}$$

The rightmost form is the result of the cancelation of two oppositely signed terms of the form $\tilde{H}\tilde{\mathtt{X}}^i\tilde{H}$. To work out the above we use the square of the Hamiltonian operator.

$$\tilde{H}^2 \;=\; -\,\hbar^2\frac{\partial^2}{\partial t^2}\psi \;=\; \Big(\,-\,\hbar^2c^2\nabla^2\psi \;+\; m^2c^4\,\Big)\psi \tag{10.46}$$

The mass term m^2c^4 commutes with the position operator, so it does not contribute. The non-commuting terms we need are the second order derivatives over the axis x^i corresponding with the i-th component of the position operator $\tilde{\mathtt{X}}^i$.

$$\tilde{\mathtt{A}}^i \;=\; -\,\frac{1}{\hbar^2}\left[\,\frac{i\hbar}{2mc}\,x^i\,\frac{\partial}{\partial x^o},-\hbar^2c^2\left(\frac{\partial}{\partial x^i}\right)^2\,\right] \;=\; -\frac{i\hbar c}{m}\,\frac{\partial}{\partial x^o}\,\frac{\partial}{\partial x^i} \tag{10.47}$$

When we compare this with (10.32) for the velocity operator $\tilde{\mathtt{V}}^i$ we see an extra time derivative. This is what we would expect from the classical case.

Meaning of the acceleration operator

If we apply the operator $\tilde{\mathtt{A}}^i$ in the standard way on the Klein Gordon field. then we might hope to get an *acceleration-density* $\mathtt{A}^i$ in the form of.

$$\mathtt{A}^i \;\overset{?}{=}\; a^i\,\gamma\,\psi^*\psi \tag{10.48}$$

Where a^i is the local acceleration, and the factor $\gamma\,\psi^*\psi$ is the local relativistic density. Since $\psi^*\psi$ is a Lorentz scalar we need a local γ here to correct the density for the local Lorentz contraction. Compare this to the *position-density* and *velocity-density* we obtained.

$$\mathtt{X}^i \;=\; x^i\,\gamma\,\psi^*\psi \qquad\qquad \mathtt{V}^i \;=\; v^i\,\gamma\,\psi^*\psi \tag{10.49}$$

Since $v^i\gamma$ is proportional to the the momentum, the phase change rate over space, and we take the time-derivative:

$$\mathtt{A}^i \;=\; \frac{\partial}{\partial t}\Big(v^i\gamma\Big)\,\psi^*\psi \qquad \Big(\;\neq\; a^i\,\gamma\,\psi^*\psi\;\Big) \tag{10.50}$$

We do not get the average acceleration if we integrate $\mathtt{A}^i$ over all space. So what do we get? Recall the definition of the spatial components of the relativistic 4-velocity $\mathtt{U}^\mu$

$$\mathtt{U}^i \;=\; \frac{\partial x^i}{\partial \tau} \;=\; \gamma\,\frac{\partial x^i}{\partial t} \tag{10.51}$$

It is the velocity measured in the proper time of the particle. The factor γ is valid for an arbitrary accelerating particle since.

$$\tau \;=\; \int_0^t \frac{1}{\gamma}dt \qquad \Longrightarrow \qquad \frac{\partial \tau}{\partial t} \;=\; \frac{1}{\gamma} \tag{10.52}$$

The relativistic four-acceleration is the second derivative of x^i in regard with the proper time τ.

$$\frac{\partial\,\mathtt{U}^i}{\partial\,\tau} \;=\; \frac{\partial\,\mathtt{U}^i}{\partial\,t}\frac{\partial\,\tau}{\partial\,t} \;=\; \frac{\partial\,\mathtt{U}^i}{\partial\,t}\frac{1}{\gamma} \tag{10.53}$$

We now can see that $\mathtt{A}^i$ is the product of the relativistic density $\gamma\,\psi^*\psi$ and the relativistic four-acceleration.

$$\mathtt{A}^i \;=\; \frac{\partial^2 x^i}{\partial\,\tau^2}\;\gamma\,\psi^*\psi \tag{10.54}$$

Which means that the integration over all space of $\mathtt{A}^i$ gives us the average relativistic four-acceleration.

$$\left.\frac{\partial^2 x^i}{\partial\,\tau^2}\right|_{avg} \;=\; \int\;\mathtt{A}^i\;dx^3 \;=\; \int\;\frac{\partial\gamma v^i}{\partial\,t}\,\psi^*\psi\;dx^3 \tag{10.55}$$

Application of the acceleration operator

If we express the Klein Gordon field ψ in a way which separates the phase ϕ and its amplitude, and furthermore define related fields $\hat{\psi}$ and $|\psi|^2$ as.

$$\begin{array}{rcl} \psi & = & \exp\left(\, a + i\phi \,\right) \\ \hat{\psi} & = & \exp\left(\, i\phi \,\right) \\ |\psi|^2 & = & \exp\left(\, 2a \,\right) \end{array} \tag{10.56}$$

Then we can apply the operator $\tilde{\mathcal{O}}^i$, operating on the phase only, in the following way.

$$\mathcal{O}^i \;\;=\;\; \left(\, \tilde{\mathcal{O}}^{i\,*}\; \hat{\psi} \,\right)\; |\psi|^2 \tag{10.57}$$

As long as $\tilde{\mathcal{O}}^i$ contains no higher as first order derivatives then we can write this in the usual asymmetric form.

$$\left(\, \tilde{\mathcal{O}}^i\; \hat{\psi} \,\right)\; |\psi|^2 \;=\; \frac{1}{2}\left(\, \psi^* \tilde{\mathcal{O}}^i \psi \;-\; \psi\, \tilde{\mathcal{O}}^i \psi^* \,\right) \tag{10.58}$$

This won't work however if the operator contains higher order derivatives as is the case of the acceleration operator. When we apply the operator $\tilde{\mathtt{A}}^i$ in the usual anti-symmetric way on the Klein Gordon field ψ,

$$\mathtt{A}^i \;\;\neq\;\; -\frac{i\hbar c}{m}\left(\, \psi^* \frac{\partial}{\partial x^o}\frac{\partial}{\partial x^i}\psi \;-\; \psi \frac{\partial}{\partial x^o}\frac{\partial}{\partial x^i}\psi^* \,\right) \tag{10.59}$$

then we get the amplitude component a mixed in the result. The application of $\tilde{\mathtt{A}}^i$ that correctly does what we want is given by evaluating (10.50).

$$\begin{aligned} \mathtt{A}^i \;&=\; \frac{\partial}{\partial\, t}\left(\, v^i \gamma \,\right)\; \psi^* \psi \\ &=\; -\,\frac{i\hbar}{2m}\left[\, \frac{\partial}{\partial t}\left(\, \frac{1}{\psi}\frac{\partial \psi}{\partial x^i} - \frac{1}{\psi^*}\frac{\partial \psi^*}{\partial x^i} \,\right)\right] \psi^* \psi \end{aligned} \tag{10.60}$$

$$= \;\; -\,\frac{i\hbar}{2m}\left(\, \psi^* \frac{\partial}{\partial t}\frac{\partial}{\partial x^i}\psi \;-\; \psi\, \frac{\partial}{\partial t}\frac{\partial}{\partial x^i}\psi^* \;-\; \frac{\psi^*}{\psi}\frac{\partial \psi}{\partial t}\frac{\partial \psi}{\partial x^i} \;+\; \frac{\psi}{\psi^*}\frac{\partial \psi^*}{\partial t}\frac{\partial \psi^*}{\partial x^i} \,\right)$$

This more cumbersome expression has two additional terms which cancel the unwanted terms which are depending on the amplitude. It can be checked, by substituting $\psi = \exp(a + i\phi)$ that this expression is equivalent to,

$$\mathtt{A}^i \quad = \quad -\frac{i\hbar}{m}\left(\frac{\partial}{\partial t}\frac{\partial}{\partial x^i}\,\hat{\psi}\right)\psi^*\psi \qquad (10.61)$$

as in equation (10.57), where $\hat{\psi} = \exp(i\phi)$. Our confidence in (10.60) is boosted considerably because it can be used to derive the acceleration from the electro magnetic fields. (as will be shown briefly in the next section). Working out (10.61) gives us.

$$\mathtt{A}^i \quad = \quad -\frac{i\hbar}{m}\left(\frac{\partial}{\partial t}\frac{\partial\phi}{\partial x^i}\right)\psi^*\psi \quad = \quad \frac{1}{m}\left(\frac{\partial p^i}{\partial t}\right)\psi^*\psi \qquad (10.62)$$

The result does not depend on the order in which the derivatives are applied, but note that on exchanging ∂x^i and ∂t we get.

$$\mathtt{A}^i \quad = \quad -\frac{i\hbar}{m}\left(\frac{\partial}{\partial x^i}\frac{\partial\phi}{\partial t}\right)\psi^*\psi \quad = \quad -\frac{1}{m}\left(\frac{\partial p^o}{\partial x^i}\right)\psi^*\psi \qquad (10.63)$$

Because we associate $\partial\phi/\partial x^i$ with the momentum p^i but we associate $\partial\phi/\partial t$ with the energy p^o instead. The phase ϕ is a scalar function of the four component p^μ which leads to a redundancies.

The above shows that there are two terms which contribute identically to ϕ and thus to $\mathtt{A}^i$ also. We have to consider them both. Since the two expressions are equal we can write.

$$\frac{\partial p^i}{\partial t} \quad = \quad -\frac{\partial p^o}{\partial x^i} \qquad (10.64)$$

This tells us that there will be an acceleration in the direction of x^i if the (potential) energy decreases along x^i, which can for instance be the case in classical general relativity near a massive body.

In this case the field is an interacting field and the potential difference should actually be described by an interaction term with gravitational field.

10.8 Acceleration operator with EM interaction

The chapter on the "*Lorentz force derived from Klein Gordon's equation*" handles the electromagnetic interaction with the Klein Gordon field in great detail. Here we will briefly preview how the *4-acceleration density operator* we have just arrived leads to the electromagnetic Lorentz force.

The four-potential A^μ determines the electromagnetic interaction. Multiple like charges brought close to each other increase each others energy (phase change rate), while multiple charges moving in a bunch increase each other's momentum (spatial phase change rate).

The total phase change rates determines the total four-momentum, the so called *canonical* momentum. This is the sum of the *inertial* momentum (from the mass) plus the *interaction* momentum. The physical velocity is still determined by the inertial momentum. The effect of the electromagnetic four-potential is an additional phase change added to the field when propagating.

If A^μ is zero and we increase it then it doesn't simply adds phase change rates to the field. Rather the field will change it's inertial momentum to *compensate* the added phase change rates. The simple reason is special relativity. The phase of the field at a certain point would depend on the spatial integral of the phase rates. If A^μ changes at an arbitrary far away point then the phase would change anywhere *instantaneously*. Hence, inertial momentum has to change in order to compensate the induced phase change rates from the four-potential.

The derivatives ∂_μ determine the total canonical momentum. We use the *gauge covariant derivative* D_μ, which removes the electromagnetic interaction to recover the inertial momentum which determines the actual physical speed

$$D_\mu = \partial_\mu + ieA_\mu/\hbar \tag{10.65}$$

We therefor replace the normal derivatives in our expression for the four-acceleration density (10.60) with the covariant ones.

$$\frac{\partial}{\partial x^o} \implies \frac{\partial}{\partial x^o} + i\frac{e}{\hbar}A^o, \qquad \frac{\partial}{\partial x^i} \implies \frac{\partial}{\partial x^i} - i\frac{e}{\hbar}A^i \tag{10.66}$$

Working this out step by step gives us the following expression.

$$\mathtt{A}^i \;=\; -\frac{ihc}{2m}\Bigg(\tag{10.67}$$

$$\begin{array}{llll}
+\ \psi^* \dfrac{\partial}{\partial x^o}\dfrac{\partial}{\partial x^i}\psi & -\ i\dfrac{e}{\hbar}\psi^*\dfrac{\partial A^i\psi}{\partial x^o} & +\ i\dfrac{e}{\hbar}\psi^* A^o \dfrac{\partial\psi}{\partial x^i} & +\ \dfrac{e^2}{\hbar^2}A^oA^i\,\psi^*\psi \\
-\ \psi\ \dfrac{\partial}{\partial x^o}\dfrac{\partial}{\partial x^i}\psi^* & +\ i\dfrac{e}{\hbar}\psi\ \dfrac{\partial A^i\psi^*}{\partial x^o} & -\ i\dfrac{e}{\hbar}\psi\ A^o\dfrac{\partial\psi^*}{\partial x^i} & -\ \dfrac{e^2}{\hbar^2}A^oA^i\,\psi^*\psi \\
-\ \psi^*\dfrac{\partial\psi}{\partial x^o}\dfrac{\partial\psi}{\partial x^i}\psi^{-1} & +\ i\dfrac{e}{\hbar}\psi^* A^i\dfrac{\partial\psi}{\partial x^o} & -\ i\dfrac{e}{\hbar}\psi^* A^o\dfrac{\partial\psi}{\partial x^i} & -\ \dfrac{e^2}{\hbar^2}A^oA^i\,\psi^*\psi \\
+\ \psi\ \dfrac{\partial\psi^*}{\partial x^o}\dfrac{\partial\psi^*}{\partial x^i}\psi^{-*} & -\ i\dfrac{e}{\hbar}\psi\ A^i\dfrac{\partial\psi^*}{\partial x^o} & +\ i\dfrac{e}{\hbar}\psi\ A^o\dfrac{\partial\psi^*}{\partial x^i} & +\ \dfrac{e^2}{\hbar^2}A^oA^i\,\psi^*\psi\ \Bigg)
\end{array}$$

The first column here is simply (10.60) without the interaction. The fourth column adds up to zero and the third column adds up to zero as well. The second column contains the difference between interaction and no interaction.

$$-\frac{e}{m}\frac{\partial A^i}{\partial x^o}\,\psi^*\psi \tag{10.68}$$

If we express the total canonical momentum by capitol P^μ then the interaction version of (10.62) becomes simply.

$$+\frac{\partial P^i}{\partial x^o} \;=\; \left(+\frac{\partial p^i}{\partial x^o}\;+\;e\,\frac{\partial A^i}{\partial x^o}\right) \tag{10.69}$$

It doesn't make any difference if we change the order of the two derivatives. The interacting version of expression (10.63) becomes.

$$-\frac{\partial P^o}{\partial x^i} \;=\; \left(-\frac{\partial p^o}{\partial x^i}\;-\;e\,\frac{\partial A^o}{\partial x^i}\right) \tag{10.70}$$

The two expressions are degenerate with respect to each other since ψ is a scalar function determined by two vectors p^μ and A^μ which determine the phase ϕ. The represent two causes with the same outcome and we have to consider them both.

Equating the two gives us the expression for the four-acceleration due to electromagnetic interaction.

$$\frac{\partial p^i}{\partial x^o} \quad = \quad - \ \frac{\partial p^o}{\partial x^i} \ + \ e\left(-\frac{\partial A^i}{\partial x^o} - \frac{\partial A^o}{\partial x^i}\right) \tag{10.71}$$

The last term contains the definition of the Electric field E^i so that we can write.

$$\frac{\partial p^x}{\partial x^o} \quad = \quad - \ \frac{\partial p^o}{\partial x^i} \ + \ \frac{e}{c}\,\mathsf{E}^i \tag{10.72}$$

We have found the Electric part of the Lorentz force. We obtain the Magnetic part if we consider the phase changes at a moving point instead of a point at rest. This is handled in great detail and with lots of illustrating images in the chapter *"Lorentz force derived from Klein Gordon's equation"*

10.9 The angular momentum operator

The angular momentum operator should extract the quantum mechanical equivalent of the classical $\vec{r} \times \vec{p}$. To get the *density* of the angular momentum we need the *density* of the momentum.

More to come here...